

CONSTRUCTING THE DEMAND FUNCTION OF A STRICTLY CONVEX PREFERENCE RELATION

MATTHEW HENDTLASS

ABSTRACT. We give conditions under which the demand function of a strictly convex preference relation can be constructed.

INTRODUCTION

This paper gives conditions under which the demand function of a strictly convex preference relation can be constructed, and should be seen as a continuation of the work of Douglas Bridges [4, 5, 6, 8] to examine aspects of mathematical economics in a rigorously constructive manner, see also [12]. In particular, Bridges considered the problem that we consider here in [6]. Corollary 12 is a generalisation of the main result of [6] and our proof, although less elegant, is also somewhat simpler.

Following Bridges we take, as our starting point, the standard configuration in microeconomics consisting of a consumer whose consumption set X is a compact, convex subset of \mathbf{R}^n ordered by a strictly ordered preference relation \succ . For a given price vector $p \in \mathbf{R}^n$ and a given initial endowment w , the consumers *budget set*

$$\beta(p, w) = \{x \in X : p \cdot x \leq w\}$$

is the collection of all consumption bundles available to the consumer.

As detailed in [6], it is easy to show that classically, if $\beta(p, w) \neq \emptyset$, then there exists a unique \succ -maximal point $\xi_{p,w} \in \beta(p, w)$: $\xi_{p,w} \succ x$ for all $x \in \beta(p, w)$. Let T be the set of pairs consisting of a price vector p and an initial endowment w for which $\beta(p, w)$ is inhabited. If the preference relation \succ is continuous, then a sequential compactness argument gives the sequential, and hence pointwise, continuity of the *demand function* F on T which sends (p, w) to the maximal element $\xi_{p,w}$ of $\beta(p, w)$ (see, for example, chapter 2, section D of [16]).

Bridges asked under what conditions can we

1. Compute the demand function F ;
2. Compute a modulus of uniform continuity for F : given $\varepsilon > 0$, can we produce $\delta > 0$ such that if $(p, w), (p', w') \in T$ with $\|(p, w) - (p', w')\| < \delta$, then $\|F(p, w) - F(p', w')\| < \varepsilon$.

In [6] Bridges introduced the notion of a uniformly rotund preference relation and showed that if \succ is uniformly rotund and you restrict F to a compact subset of T on which the consumer cannot be satiated, then F is uniformly continuous. Theorem 12 shows that we do not need the hypothesis that our consumer is nonsatiated. Theorems 1 and 9 encapsulate what we can say about strictly convex preference relations, which is more than one might think.

We work in Bishop's style constructive mathematics. Any proof in this framework embodies an algorithm, so when we show that there exists x such that $P(x)$, our proof gives an explicit construction of an object x together with a proof that $P(x)$ holds. Formally we take Bishop's constructive mathematics to be Aczel's constructive Zermelo-Fraenkel set theory (**CZF**) with intuitionistic logic and the axiom of dependent choice [2]. By interpreting **CZF** in Martin-Löf type theory [1], the

algorithmic nature of our proofs can be made explicit; and techniques for programme extraction from such proofs have been well studied [15]. We direct the reader to [3, 10] for an introduction to the practice of Bishop's constructive mathematics, and to [4, 5] for an introduction to Bridges' programme to constructivise mathematical economics.

A *preference relation* \succ on a set X is a binary relation which is

- asymmetric: if $x \succ y$, then $\neg(y \succ x)$;
- negatively transitive: if $x \succ y$, then for all z either $x \succ z$ or $z \succ y$.

If $x \succ y$, we say that x is *preferable* to y . We write $x \succcurlyeq y$, x is *preferable or indifferent* to y , for $\neg(y \succ x)$. We note that $x \succ x$ is contradictory, that \succ and \succcurlyeq are transitive, and that if either $x \succcurlyeq y \succcurlyeq z$ or $x \succ y \succcurlyeq z$, then $x \succcurlyeq z$.

Let \succ be a preference relation on a subset X of \mathbf{R}^N .

- \succ is a *continuous preference relation* if the graph

$$\{(x, x') : x \succ x'\}$$

of \succ is open.

- \succ is *strictly convex* if X is convex and $tx + (1 - t)x' \succ x$ or $tx + (1 - t)x' \succ x'$ whenever $x, x' \in X$ with $x \neq x'$ and $t \in (0, 1)$.
- X is *uniformly rotund* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, x' \in X$, if $\|x - x'\| \geq \varepsilon$, then

$$\left\{ \frac{1}{2}(x + x') + z : z \in B(0, \delta) \right\} \subset X,$$

where $B(x, r)$ is the open ball of radius r centred on x . The preference relation \succ is *uniformly rotund* if X is uniformly rotund and for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - x'\| \geq \varepsilon$ ($x, x' \in X$), then for each $z \in B(0, \delta)$ either $\frac{1}{2}(x + x') + z \succ x$ or $\frac{1}{2}(x + x') + z \succ x'$.

A uniformly rotund preference relation is strictly convex.

A set S is said to be *inhabited* if there exists x such that $x \in S$. An inhabited subset S of a metric space X is *located* if for each $x \in X$ the *distance*

$$\rho(x, S) = \inf \{ \rho(x, s) : s \in S \}$$

from x to S exists. If X is located and its *metric complement*

$$-X = \{x \in \mathbf{R}^n : \rho(x, X) > 0\}$$

is also located, then X is said to be *bilocated*. An ε -*approximation* to S is a subset T of S such that for each $s \in S$, there exists $t \in T$ such that $\rho(s, t) < \varepsilon$. We say that S is *totally bounded* if for each $\varepsilon > 0$ there exists a finitely enumerable¹ ε -approximation to S ; totally bounded sets are located. A metric space X is *compact* if it is complete and totally bounded. We will use $\|\cdot\|$ to represent the Euclidean norm, $\|\cdot\|_1$ for the norm $x \mapsto \sum_{i=1}^n x_i$ on \mathbf{R}^n , and we write ρ, ρ_1 for the respective induced metrics.

¹A set is *finitely enumerable* if it is the image of $\{1, \dots, n\}$ for some $n \in \mathbf{N}$, and a set is *finite* if it is in bijection with $\{1, \dots, n\}$ for some $n \in \mathbf{N}$; constructively these notions are distinct.

CONSTRUCTING MAXIMA

In this section we focus on the construction of maximally preferred elements of a consumption set X . Our main result is

Theorem 1. *Let \succ be a continuous, strictly convex preference relation on an inhabited, compact subset X of Euclidean space. Then there exists a unique $\xi \in X$ such that $\xi \succcurlyeq x$ for all $x \in X$.*

Our proof proceeds by induction. The following lemma provides the key to proving the one dimensional case.

Lemma 2. *Let \succ be a strictly convex preference relation on $[0, 1]$. Then either $1/2 \succcurlyeq x$ for all $x \in [0, 1/4)$ or $1/2 \succcurlyeq x$ for all $x \in (3/4, 1]$.*

Proof. Applying the strict convexity of \succ to $1/4 \in (0, 3/4)$, $1/2 \in (1/4, 3/4)$, $3/4 \in (1/2, 1)$ yields

$$\begin{aligned} 1/4 \succ 0 \quad \text{or} \quad 1/4 \succ 3/4; \\ 1/2 \succ 1/4 \quad \text{or} \quad 1/2 \succ 3/4; \\ 3/4 \succ 1/4 \quad \text{or} \quad 3/4 \succ 1. \end{aligned}$$

It follows that either $1/2 \succ 1/4 \succ 0$ or $1/2 \succ 3/4 \succ 1$. In the first case suppose that $x \succcurlyeq 1/2$ for some $x \in [0, 1/4)$. Then, by the strict convexity and transitivity of \succ , $1/4 \succ 1/2$; this contradiction ensures that $1/2 \succcurlyeq x$ for all $x \in [0, 1/4)$. Similarly, in the second case $1/2 \succcurlyeq x$ for all $x \in (3/4, 1]$. \square

Lemma 3. *If \succ is a strictly convex, continuous preference relation on $[0, 1]$, then there exists $\xi \in [0, 1]$ such that $\xi \succcurlyeq x$ for all $x \in [0, 1]$.*

Proof. We inductively construct intervals $[\underline{\xi}_n, \bar{\xi}_n]$ such that, for each n ,

1. $[\underline{\xi}_n, \bar{\xi}_n] \subset [\underline{\xi}_{n-1}, \bar{\xi}_{n-1}]$;
2. $\bar{\xi}_n - \underline{\xi}_n = (4/5)^n$;
3. for each $x \in [0, 1] \setminus [\underline{\xi}_n, \bar{\xi}_n]$, there exists $y \in [\underline{\xi}_n, \bar{\xi}_n]$ such that $y \succcurlyeq x$.

To begin the construction set $\underline{\xi}_0 = 0$ and $\bar{\xi}_0 = 1$. At stage n , rescaling for $n > 1$, we apply Lemma 2; if the first case obtains, then we set $\underline{\xi}_n = (3\underline{\xi}_{n-1} + \bar{\xi}_{n-1})/4$ and $\bar{\xi}_n = \bar{\xi}_{n-1}$. In the second case we set $\underline{\xi}_n = \underline{\xi}_{n-1}$ and $\bar{\xi}_n = (\underline{\xi}_{n-1} + 3\bar{\xi}_{n-1})/4$. By the transitivity of \succcurlyeq , we need only check condition 3. for $[\underline{\xi}_{n-1}, \bar{\xi}_{n-1}] \setminus [\underline{\xi}_n, \bar{\xi}_n]$, and by Lemma 2 $y = (\underline{\xi}_{n-1} + \bar{\xi}_{n-1})/2$ suffices for each such point.

Let ξ be the unique intersection of the $[\underline{\xi}_n, \bar{\xi}_n]$. Since \succcurlyeq is continuous, the maximality of ξ follows from 3. \square

Lemma 4. *If \succ is a strictly convex, continuous preference relation on $[a, b]$, where $a \leq b$, then there exists $\xi \in [a, b]$ such that $\xi \succcurlyeq x$ for all $x \in [a, b]$.*

Proof. Construct an increasing binary sequence $(\lambda_n)_{n \geq 1}$ such that

$$\begin{aligned} \lambda_n = 0 &\Rightarrow b - a < 1/n; \\ \lambda_n = 1 &\Rightarrow b - a > 1/(n+1). \end{aligned}$$

Without loss of generality, we may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $x_n = a$ and if $\lambda_n = 1 - \lambda_{n-1}$, then we apply Lemma 3, after some scaling, to construct a \succ -maximal element x in $[a, b]$, and set $x_k = x$ for all $k \geq n$. Then for $m > n$, $|x_n - x_m| < 2/(n-1)$, so $(x_n)_{n \geq 1}$ converges to some element $\xi \in [a, b]$. If there exists $x \neq \xi$ such that $x \succ \xi$, then $b - a > 0$ and we get a contradiction to Lemma 3. The result now follows from continuity. \square

We use π_i to denote the i -th projection function, and we write $[x, y]$ for

$$\{tx + (1 - t)y : t \in [0, 1]\}.$$

Here is the **proof of Theorem 1**:

Proof. We proceed by induction on the dimension n of the space containing X . Lemma 4 is just the case $n = 1$. Now suppose we have proved the result for n and consider a strictly convex preference relation \succ on a compact, convex subset X of \mathbf{R}^n . Define a preference relation \succ' on $\pi_1(X) = [a, b]$ by

$$s \succ_i t \iff \exists x \in X \forall y \in X (\pi_1(x) = s \text{ and if } \pi_1(y) = t, \text{ then } x \succ y).$$

Then \succ' is strictly convex and sequentially continuous: let $s_1, s_2, t \in [a, b]$ with $s_1 < t < s_2$. By the induction hypothesis there exist ξ_1, ξ_2 such that $\pi_1(\xi_i) = s_i$ and $\xi_i \succ x$ for all $x \in X$ with $\pi_1(x) = s_i$ ($i = 1, 2$). Let z be the unique element of $[\xi_1, \xi_2]$ such that $\pi_1(z) = t$. Then, by the strict convexity of \succ , either $z \succ \xi_1$ or $z \succ \xi_2$. In the first case $t \succ' s_1$ and in the second $t \succ' s_2$. Hence \succ is strictly convex. That \succ' is continuous is straightforward.

We can now apply Lemma 4 to construct a maximal element ξ_1 of $(\pi_1(X), \succ')$, and then the induction hypothesis to construct a maximal element of

$$S = \{x \in X : \pi_1(x) = \xi_1\}$$

with $\succ|_S$. Clearly $\xi = \xi_1 \times \xi_2$ is a \succ -maximal element of X . The uniqueness of maximal elements follows directly from the strict convexity of \succ . \square

We shall have need for the following simple corollary, which is of independent interest.

Corollary 5. *Under the conditions of Theorem 1, if $x \in X$ and $x \neq \xi$, then $\xi \succ x$.*

Proof. Let $y = (x + \xi)/2$. Then either $y \succ x$ or $y \succ \xi$. Since $\xi \succ y$ the former must attain, so $\xi \succ y \succ x$. \square

If we are not interested in uniqueness of maxima, then we might suppose that \succ only satisfies the weaker condition of being *convex*: for all $x, y \in X$ and each $t \in [0, 1]$, either $(x + y)/2 \succ x$ or $(x + y)/2 \succ y$. We give a Brouwerian counterexample² to show that this condition is not strong enough to allow the construction of a maximal point. Let $x \in (-1/4, 1/4)$ and let $f : [0, 1] \rightarrow \mathbf{R}$ be the function given by

$$f(t) = \begin{cases} \text{sign}(x)(t - x \vee 0) & t \in [0, x \vee 0] \\ 0 & t \in [x \vee 0, 1 - x \vee 0] \\ -\text{sign}(x)(t - x \vee 0) & t \in [1 - x \vee 0, 1], \end{cases}$$

where³

$$\text{sign}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0. \end{cases}$$

Define a preference relation \succ on $[0, 1]$ by

$$t \succ s \iff f(t) > f(s).$$

²A Brouwerian counterexample is a weak counterexample: it is not an example contradicting a proposition, but an example showing a proposition to imply a principle which is unacceptable in constructive mathematics. Generally these can be considered as unprovability results.

³This is just convenient notation: formally sign is not a constructively well defined function, but the function f does exist constructively.

It is easy to see that \succ is continuous and convex. Further, if $x > 0$, then 0 is the unique maximal element, and if $x < 0$, then 1 is the unique maximal element. Now suppose that we can construct $\xi \in [0, 1]$ such that $\xi \succ t$ for all $t \in [0, 1]$; either $\xi > 0$ or $\xi < 1$. In the first case we have $\neg(x > 0)$ and in the second $\neg(x < 0)$, so the statement

‘Every continuous, convex preference relation on $[0, 1]$ has a maximal element’
 implies $\forall_{x \in \mathbf{R}}(x \leq 0 \vee x \geq 0)$, which is equivalent to the constructively unacceptable lesser limit principle of omniscience [9].

CONTINUOUS DEMAND FUNCTIONS

We now consider a consumer whose consumption set X is a closed convex subset of \mathbf{R}^n ordered by a strictly convex preference relation \succ , and who has an initial endowment $w \in \mathbf{R}$. For a given price vector $p \in \mathbf{R}$, a consumers *budget set*

$$\beta(p, w) = \{x \in X : p \cdot x \leq w\}$$

is the collection of commodity bundles the consumer can afford. The collection of maximal elements of $\beta(p, w)$ form the consumers *demand set* for price p and initial endowment w .

Lemma 6. *If $p > 0$ and there exists $x \in X$ such that $p \cdot x \leq w$, then $\beta(p, w)$ is compact and convex.*

Proof. Convexity is clear. See [6] for a proof that $\beta(p, w)$ is compact. \square

We use ∂S to denote the boundary of a subset S of some metric space.

Lemma 7. *The boundary of $\beta(p, w)$ is compact.*

Proof. If X is colocated, then $\rho(x, \partial X) = \max\{\rho(x, X), \rho(x, -X)\}$ and hence the boundary of X is located. Therefore it suffices to show that $-\beta(p, w)$ is located. This is similar to the proof of Lemma 6. \square

It now follows from Theorem 1 that the function F , the consumers *demand function*, that maps (p, w) , where p is a price vector and w an initial endowment, to the unique maximal element of $\beta(p, w)$, is well defined. By logical considerations we have that any function which can be proven to exist within Bishop’s constructive mathematics alone is classically continuous, so the consumers demand function is continuous in the classical setting.

We seek conditions under which F is constructively continuous. A function on a locally compact space is said to be *Bishop continuous* if it is pointwise continuous, and is further uniformly continuous on every compact space. Since the uniform continuity theorem—every continuous function on a compact space is uniformly continuous—is not provable in Bishop’s constructive mathematics, this is the natural notion of continuity for us to consider. We study the continuity of F by looking at the map Γ , on the set T of all inhabited $\beta(p, w)$, taking $\beta(p, w)$ to $F(p, w)$. We give T the Hausdorff metric: for located subsets A, B of a metric space Y

$$\rho_H(A, B) = \max\{\sup\{\rho(a, B) : a \in A\}, \sup\{\rho(b, A) : b \in B\}\}.$$

Our next lemma shows how studying Γ allows us to show the continuity of F .

Lemma 8. *If Γ is continuous, then F is continuous. If Γ is uniformly continuous, then for each $p \in \mathbf{R}^n$, $w \mapsto F(p, w)$ is uniformly continuous, and for each $w \in \mathbf{R}$, $p \mapsto F(p, w)$ is Bishop continuous.*

In constructive mathematics, the uniform continuity theorem—every pointwise continuous function with compact domain is uniformly continuous—is closely related to the ‘semi-constructive’ fan theorem isolated by Brouwer. In the appendix we introduce Brouwer’s fan theorem (**FT**) and the notion of a weakly (uniformly) continuous predicate, and we give a version of the uniform continuity theorem for these predicates. Our next result says that adopting Brouwer’s fan theorem is sufficient to prove the classical result that F is continuous when \succ is continuous and strictly convex. We observe that if $\beta(p, w)$ is inhabited and every component of p is positive, then

$$\beta(p, w) = \left\{ x \in \mathbf{R}^n : \sum_{i=1}^n p_i x_i \leq w \right\}$$

is a diamond, and if the *diameter*

$$\sup\{\rho(x, y) : x, y \in \beta(p, w)\}$$

of $\beta(p, w)$ is positive, then $\beta(p, w)$ has inhabited interior.

Theorem 9. *Suppose Brouwer’s full fan theorem holds. If \succ is continuous and strictly convex, then F is Bishop continuous.*

Proof. Since **FT** implies that every continuous function on a compact space is uniformly continuous, it suffices, by Lemma 8, to show that Γ is continuous. Fix $\varepsilon > 0$, and $(p, w) \in \mathbf{R}^{n+1}$ such that $\beta(p, w)$ is inhabited; we write $S = \beta(p, w)$ and $\xi = F(p, w)$.

Either $\rho(\xi, \partial S) > 0$ or $\rho(\xi, \partial S) < \varepsilon/2$. In the first case ξ is maximal on the entire set of consumer bundles, so it suffices to set $\delta = \varepsilon$. In the second case, let φ be the natural bijection of $[0, 1]^n$ with $T \equiv \partial\beta(p, w) \setminus B_{\rho_1}(x, \varepsilon/2)$; without loss of generality, φ is nonexpansive. We define a predicate on $[0, 1]$ by

$$P(x, \alpha, \delta) \Leftrightarrow \forall_{y \in B(\varphi(x), \delta)} \xi \succ y.$$

Then P is a weakly continuous predicate: condition (i) follows from Corollary 5 and the lower pointwise continuity of \succ ; condition (ii) follows from elementary geometry, given that φ is nonexpansive. By Theorem 17, P is weakly uniformly continuous and hence there exists $\delta > 0$ such that every $y \in B(x, \delta)$ is strictly less preferable than ξ for all $x \in T$. If $\rho(x, S) < \min\{\delta, \varepsilon\}/2$, then $\rho(x, T) < \delta$, $x \in S$, or $x \in B(\xi, \varepsilon)$. In the first two cases $\xi \succ x$; it follows that $F(p', w') \in B(\xi, \varepsilon)$ whenever $\rho_H(\beta(p, w), \beta(p', w')) < \min\{\delta, \varepsilon\}/2$. \square

It may seem a little odd that we choose to work in Bishop’s constructive mathematics because we are interested in producing results with computational meaning, but that we then add an extra principle **FT** to our framework. In particular, the inconsistency of Brouwer’s fan theorem with recursive analysis [9] may cause some consternation. The constructive nature of the fan theorem can be intuitively justified as follows: in order to assert that B is a bar we must have a proof that B is a bar, and a proof is a finite object; therefore an examination of the finite information used in the proof that B is a bar should reveal the uniform bound that the fan theorem gives us. Although this argument does not hold up in Bishop’s constructive mathematics, if your objects are presented in the right way (and indeed a very nature way from a computational point of view), then the fan theorem can be proved [11, 17].

We pause here to give a consequence of Theorem 9. Consider a system with N commodities, n producers, and m consumers. To each producer we associate a production set $Y_i \subset \mathbf{R}^N$; and to each consumer a consumption set $X_i \subset \mathbf{R}^N$ endowed with a preference relation \succ_i . Further we assume that each consumer has no initial endowment. A *competitive equilibrium* of an economy consists of a price vector $\mathbf{p} \in \mathbf{R}^N$, points $\xi_1, \dots, \xi_i \in \mathbf{R}^N$, and a vector η in the *aggregate production set*

$$Y = Y_1 + \dots + Y_n,$$

satisfying

- E1** $\xi_i \in D_i(\mathbf{p})$ for each $1 \leq i \leq m$.
- E2** $\mathbf{p} \cdot \mathbf{y} \leq \mathbf{p} \cdot \eta = 0$ for all $\mathbf{y} \in Y$.
- E3** $\sum_{i=1}^m \xi_i = \eta$.

An economy is said to have *approximate competitive equilibria* if for all $\varepsilon > 0$ there exist a price vector $\mathbf{p} \in \mathbf{R}^N$, points $\xi_1, \dots, \xi_i \in \mathbf{R}^N$, and a vector η satisfying **E1**, **E3**, and

$$\mathbf{AE} \quad \mathbf{p} \cdot \eta > -\varepsilon.$$

The work in [12] together with Theorem 9 gives the next result, which is an approximate version of McKenzie's theorem on the existence of competitive equilibria [14].

Theorem 10. *Assume that Brouwer's fan theorem holds. Suppose that*

- (i) *each X_i is compact and convex;*
- (ii) *each \succ_i is continuous and strictly convex;*
- (iii) *$(X_i \cap Y)^\circ$ is inhabited for each i ;*
- (iv) *Y is a located closed convex cone;*
- (v) *$Y \cap \{(x_1, \dots, x_N) : x_i \geq 0 \text{ for each } i\} = \{0\}$; and*
- (vi) *for each $\mathbf{p} \in \mathbf{R}^N$ and each i , if $\sum_{i=1}^m F_i(\mathbf{p}) \in Y$, then there exists $\mathbf{x}_i \in X_i$ such that $\mathbf{x}_i \succ_i F_i(\mathbf{p})$.*

Then there are approximate competitive equilibria.

Uniformly rotund preference relations. In order to prove Theorem 9 we effectively strengthened our theory, and therefore weakened our notion of computable. The other natural approach toward proving the existence of a Bishop continuous demand function is to strengthen the conditions on \succ . We follow the lead of Bridges in [6] and focus on uniformly rotund preference relations.

Hereafter, we extend the domain of Γ to all inhabited, compact, convex subsets of X . Theorem 1 still ensures that Γ is well defined.

Theorem 11. *If \succ is a uniformly rotund preference relation, then Γ is uniformly continuous.*

Proof. Let S, S' be compact, convex subsets of X and let ξ, ξ' be their \succ -maximal points. Fix $\varepsilon > 0$ and let $\delta' > 0$ be such that if $\|x - x'\| \geq \varepsilon$ ($x, x' \in X$), then for each $z \in B(0, \delta')$ either $\frac{1}{2}(x + x') + z \succ x$ or $\frac{1}{2}(x + x') + z \succ x'$, and set $\delta = \min\{\varepsilon, \delta'\}/2$.

If $\rho_H(S, S') < \delta$, then $\|\xi - \xi'\|$: Let S, S' be such that $\rho_H(S, S') < \delta$ and suppose that $\|\xi - \xi'\| > \varepsilon$. Since S, S' are convex

$$S \cap B((\xi + \xi')/2, \delta) \text{ and } S' \cap B((\xi + \xi')/2, \delta)$$

are both inhabited; let z be an element of the former set and let z' be an element of the latter. By the maximality of $\xi \in S$ and our choice of δ , $z \succ \xi'$; similarly, $z' \succ \xi$. Therefore

$$\xi \succ z \succ \xi' \succ z' \succ \xi,$$

which is absurd. Hence $\|\xi - \xi'\| \leq \varepsilon$. □

As a corollary we have the following improvement on the main result of [6].

Corollary 12. *Let \succ be a uniformly rotund preference relation on a compact, uniformly rotund subset X of \mathbf{R}^n , and let S be a subset of $\mathbf{R}^n \times \mathbf{R}$ such that $\beta(p, w)$ is inhabited for each $(p, w) \in S$. Then for each $p \in \mathbf{R}^n$, the function $w \mapsto F(p, w)$ is uniformly continuous, and for each $w \in \mathbf{R}$, the function $p \mapsto F(p, w)$ is Bishop continuous. In particular, F is Bishop Continuous.*

Proof. The result follows directly from Lemma 8 and Theorem 11. □

Not surprisingly, a less uniform version of rotundness is enough to give us the pointwise continuity of Γ . A subset X of \mathbf{R}^n is *rotund* if for each $x \in X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x' \in X$, if $\|x - x'\| \geq \varepsilon$, then

$$\left\{ \frac{1}{2}(x + x') + z : z \in B(0, \delta) \right\} \subset X.$$

A preference relation \succ is *rotund* if X is rotund and for each $x \in X, \varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - x'\| \geq \varepsilon$ ($x' \in X$), then for each $z \in B(0, \delta)$ either $\frac{1}{2}(x + x') + z \succ x$ or $\frac{1}{2}(x + x') + z \succ x'$.

Theorem 13. *If \succ is a rotund preference relation, then Γ is continuous.*

Proof. The proof is, of course very similar to the proof of Theorem 11. Let S be a compact, convex subset of X and let ξ be the unique maximal element of S . Fix $\varepsilon > 0$. Pick $\delta > 0$ such that if $\|\xi - x\| \geq \varepsilon$ ($x \in X$), then for each $z \in B(0, \delta)$ either $\frac{1}{2}(\xi + x) + z \succ \xi$ or $\frac{1}{2}(\xi + x) + z \succ x'$. If S' is a compact, convex subset of X , with maxima ξ' , such that $\rho_H(S, S') < \delta$, then the assumption that $\|\xi - \xi'\| > \varepsilon$ leads to a contradiction as in the proof of Theorem 11. \square

By the next result, Theorem 11 can be used to improve on Theorem 9.

Proposition 14. *Assume Brouwer's fan theorem. If \succ is continuous and strictly convex, then \succ is uniformly rotund.*

Proof. Without loss of generality,

$$C = \{(x, y) \in X^2 : \|x - y\| \geq \varepsilon\}$$

is compact; moreover

$$P((x, y), \varepsilon, \delta) \equiv \|x - y\| < \varepsilon \vee \forall z \in B((x + y)/2, \delta)(z \succ x \vee z \succ y)$$

defines a continuous predicate on C . Hence P is uniformly continuous by Theorem 17, but the uniformity of P says precisely that \succ is uniformly rotund. \square

Corollary 15. *Suppose Brouwer's full fan theorem holds. If \succ is continuous and strictly convex, then Γ is uniformly continuous.*

Appendix: The fan theorem and continuous predicates. Let $2^{\mathbf{N}}$ denote the space of binary sequences, Cantor's space, and let 2^* be the set of finite binary sequences. A subset S of 2^* is *decidable* if for each $a \in 2^*$ either $a \in S$ or $a \notin S$. For two elements $u = (u_1, \dots, u_m), v = (v_1, \dots, v_n) \in 2^*$ we denote by $u \frown v$ the *concatenation*

$$(u_1, \dots, u_m, v_1, \dots, v_n)$$

of u and v . For each $\alpha \in 2^{\mathbf{N}}$ and each $N \in \mathbf{N}$ we denote by $\bar{\alpha}(N)$ the finite binary sequence consisting of the first N terms of α . A set B of finite binary sequences is called a *bar* if for each $\alpha \in 2^{\mathbf{N}}$ there exists $N \in \mathbf{N}$ such that $\bar{\alpha}(N) \in B$. A bar B is said to be *uniform* if there exists $N \in \mathbf{N}$ such that for each $\alpha \in 2^{\mathbf{N}}$ there is $n \leq N$ with $\alpha(n) \in B$. The strongest form of Brouwer's fan theorem is:

FT: Every bar is uniform.

Brouwer introduced the fan theorem as a constructive principle and gave a philosophical justification for its use; it is no longer considered a valid principle of core constructive mathematics, but is still used freely by some schools (see [9]).

A predicate P on $S \times \mathbf{R}^+ \times \mathbf{R}^+$ is said to be a *continuous predicate on S* if

- (i) for each $x \in S$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that $P(x, \varepsilon, \delta)$;

(ii) if $P(x, \varepsilon, \delta)$ and $|x - y| < \delta' < \delta$, then $P(y, \varepsilon, \delta - \delta')$.

If in addition, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $P(x, \varepsilon, \delta)$ for all $x \in S$, then P is a *uniformly continuous predicate on S* .

Theorem 16. *The statement*

Every continuous predicate on $[0, 1]$ is uniformly continuous.
is equivalent to the full fan theorem.

Proof. Let P be a continuous predicate on $[0, 1]$ and fix $\varepsilon > 0$. Define a uniformly continuous function f from $2^{\mathbb{N}}$ onto $[0, 1]$ by

$$f(\alpha) = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \left(\frac{(-1)^{a_n} + 1}{2}\right),$$

where $\alpha = (a_n)_{n \geq 1}$, and let

$$B = \left\{ a \in 2^* : \forall_{x \in (f(a \smallfrown \mathbf{0}), f(a \smallfrown \mathbf{i}_1))} P\left(x, \varepsilon, (2/3)^{|a|}\right) \right\},$$

where \smallfrown denotes concatenation, $\mathbf{0} = (0, \dots)$, and $\mathbf{i}_1 = (1, 0, \dots)$. We show that B is a bar. Let $\alpha \in 2^{\mathbb{N}}$, and, using (i), pick $\delta > 0$ such that $P(f(\alpha), \varepsilon, \delta)$. Pick n such that $(2/3)^{n-1} < 2\delta$. Then

$$(f(\bar{\alpha}(n) \smallfrown \mathbf{0}), f(\bar{\alpha}(n) \smallfrown \mathbf{i}_1))_{(2/3)^n} \subset (f(\alpha) - \delta, f(\alpha) + \delta).$$

It follows from condition (ii) that $\alpha(n) \in B$; whence B is a bar.

By Brouwer's fan theorem, there exists $N > 0$ such that for all $\alpha \in 2^{\mathbb{N}}$ there is $n < N$ with $\alpha(n) \in B$. Then, by condition (ii), $P(x, \varepsilon, (2/3)^N)$ for all $x \in [0, 1]$.

Conversely, let B be a bar that is closed under extension and define a predicate P by

$$P(x, \varepsilon, \delta) \equiv \forall_x (f(x) = \alpha \rightarrow \exists_{N > 0} (2^{-N} > \delta \wedge \bar{\alpha}(N) \in B)).$$

It is easy to show that P is a pointwise continuous predicate. Hence P is uniformly continuous; in particular, there exists $\delta > 0$ such that $P(x, 1, \delta)$ holds for all $x \in [0, 1]$. Pick $N > 0$ such that $2^{-N} < \delta$. Then for all $\alpha \in 2^{\mathbb{N}}$, $\bar{\alpha}(N) \in B$. \square

Here is the result we need for the proof of Theorem 9.

Theorem 17. *Assume the fan theorem and let P be a pointwise continuous predicate on $[0, 1]^n$. Then P is a uniformly continuous predicate on $[0, 1]^n$.*

Proof. We proceed by induction on n . The case $n = 1$ is just one direction of Theorem 16. Suppose that the result holds for predicates on $[0, 1]^{n-1}$, and let P be a predicate on $[0, 1]^n$. For each x in $[0, 1]$ let P_x be the predicate on $[0, 1]$ given by

$$P_x(z, \varepsilon, \delta) \Leftrightarrow P((z, x), \varepsilon, \delta).$$

Then, since P is continuous, P_x is a continuous predicate for each $x \in [0, 1]$. It follows from our induction hypothesis that each P_x is uniformly continuous. Define a predicate P' on $[0, 1]$ by

$$P'(x, \varepsilon, \delta) \Leftrightarrow \forall_{y \in [0, 1]^{n-1}} P_x(y, \varepsilon, \delta).$$

It is easily shown that P' is a continuous predicate and that $P'(x, \varepsilon, \delta)$ holds for all $x \in [0, 1]$ if and only if $P(x, \varepsilon, \delta)$ holds for all $x \in [0, 1]^n$. By Lemma 16, P' is uniformly continuous; whence P is uniformly continuous. \square

REFERENCES

- [1] P.H.G. Aczel, ‘The type theoretic interpretation of constructive set theory’, In: MacIntyre, A. and Pacholski, L. and Paris, J, editor, *Logic Colloquium ’77*, p. 55–66, North Holland, Amsterdam, 1978.
- [2] P.H.G. Aczel and M. Rathjen, *Notes on Constructive Set Theory*, Report No. 40, Institut Mittag-Leffler, Royal Swedish Academy of Sciences, 2001.
- [3] E.A. Bishop and D.S. Bridges, ‘Constructive Analysis’, Grundlehren der Math. Wiss. 279, Springer-Verlag, Heidelberg, 1985.
- [4] D. Bridges, ‘Preference and utility: a constructive development’, *J. Math. Econom.* 9, p. 165–185, 1982.
- [5] D. Bridges, ‘The constructive theory of preference relations on a locally compact space’, *Proc. Koninklijke Nederlandse Akad. van Wetenschappen, Ser. A* 92(2), p. 141–165, 1989.
- [6] D. Bridges, ‘The construction of a continuous demand function for uniformly rotund preferences’, *Journal of Mathematical Economics* 21, p. 217–227, 1992.
- [7] D. Bridges, ‘Constructive notions of strict convexity’, *Math. Logic Quarterly* 39, p. 295–300, 1993.
- [8] D. Bridges, ‘The constructive theory of preference relations on a locally compact space—II’, *Mathematical Social Sciences* 21, p. 1–9, 1994.
- [9] D.S. Bridges and F. Richman, *Varieties of Constructive Mathematics*, London Math. Soc. Lecture Notes 97, Cambridge Univ. Press, 1987.
- [10] D.S. Bridges and L.S. Viță, *Techniques of Constructive Analysis*, Universitext, Springer-New-York, 2006.
- [11] J. Cederquist and S. Negri, ‘A constructive proof of the Heine-Borel covering theorem for formal reals’, *Types for proofs and programs*, Lecture Notes in Comput. Sci. 1158, p. 62–75, 1995.
- [12] M. Hendtlass and N. Miheisi, ‘On the construction of general equilibria in a competitive economy’, preprint.
- [13] H. Ishihara, ‘Reverse mathematics in Bishop’s constructive mathematics’, *Philosophia Scientiae, Cahier special* vol. 6, p. 43–59, 2006.
- [14] L.W. McKenzie, ‘The Classical Theorem on Existence of Competitive Equilibrium’, *Econometrica* 49(4), 819–841, 1981.
- [15] H. Schwichtenberg, ‘Program extraction in constructive analysis’, *Logicism, intuitionism, and formalism*, p. 255–275, Synth. Libr. 341, Springer, Dordrecht, 2009.
- [16] A. Takayama, ‘Mathematical Economics’, Dryden Press, Hinsdale, IL, 1974.
- [17] P. Taylor, ‘A lambda calculus for real analysis’, *Journal of Logic and Analysis* 2, 1–115, 2010.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CANTERBURY, CHRISTCHURCH 8041, NEW ZEALAND
E-mail address: `matthew.hendtlass@canterbury.ac.nz`